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On the Miyamoto-Moses Circle

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In recognition of Peter Moses' unwavering support in my triangle geometry papers, these works owe their existence to him

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ABSTRACT

The Miyamoto-Moses circle is a circle derived from a scalene acute triangle, which was introduced in mid-2023. This article showcases illustrations of this circle, certain of its properties, and depictions of other circles and triangles generated from it.

KEYWORDS: Miyamoto-Moses circle, scalene acute triangle, plane geometry, circles generated, illustrations

I. THE MIYAMOTO-MOSES CIRCLE: A JOURNEY THROUGH TIME

In July 2023, the Japanese mathematician Keita Miyamoto published problem 13035 in the "Romantics of Geometry" Facebook group. This publication introduced a novel circle configuration within scalene acute triangles. This initial discovery is presented below in theorem format, as documented in the Encyclopedia of Triangle Centers at [1].

Theorem 1.1 (Miyamoto-Moses Theorem) *In a scalene acute triangle $\triangle ABC$ with incircle γ, let A' be the midpoint of the arc BC containing A. Define B' and C' cyclically. Let Γ_a be the circle centered at A' and tangent to BC. Define Γ_b and Γ_c cyclically. Let γa be the circle internally tangent to the circumcircle at A and tangent to BC. Define γ_b and γ_c cyclically. Then, there exists a circle ω tangent to the seven circles γ, Γ_a, Γ_b, Γ_c, γ_a, γ_b, and γ_c, which is named the Miyamoto-Moses circle.*

After the presentation of this discovery, the Moroccan mathematician Er Jkh introduces a fundamental property of the Miyamoto-Moses Circle in [2].

Theorem 1.2 *The Miyamoto-Moses Circle is tangent to the Moses Circle (circle with center at the Brocard midpoint X_{39} that is tangent to the nine-point circle at the Kimberling Center X_{115})*

Remark 1.3 The Brocard Midpoint, denoted as X_{39} in reference [3], is formally[1] defined as the midpoint between the first and second Brocard points, denoted as Ω and Ω'.Furthermore, point X_{115} (see [4]) corresponds to the center of the Kiepert Hyperbola defined with detail in [5].

After conducting a thorough analysis of the circle's significance, we will now offer a visual representation. This illustration serves to clarify the shape and composition of the circle, benefiting our esteemed readership.

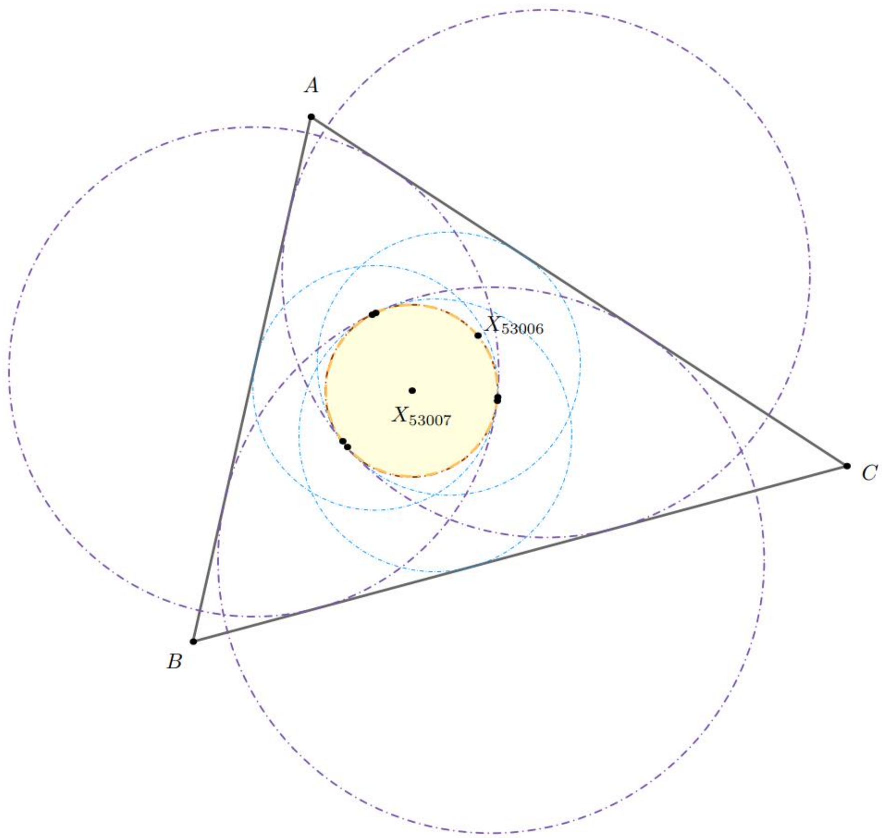

Figure 1. Diagram of the Miyamoto-Moses Circle (shaded in yellow)

In the wake of these findings, British mathematician Peter J. Moses determined the barycentric coordinates of certain triangle centers associated with the aforementioned circle (see [1]). These points will be defined in the subsequent discussion.

Definition 1.4 *The point X_{53006} is the tangency point between circle γ and the Miyamoto-Moses circle, denoted as ω.*

Definition 1.5 *The point X_{53007} is the center of the Miyamoto-Moses circle.*

As well, certain properties of triangles in perspective were encountered, which will be described and illustrated as follows.

Theorem 1.6 *Let $\varepsilon = \Gamma_a \cap \omega, \varepsilon' = \Gamma_b \cap \omega$, and $\varepsilon'' = \Gamma_c \cap \omega$ be points contained within the Miyamoto-Moses circle. Then, $\triangle \varepsilon\varepsilon'\varepsilon''$ is in perspective with the reference triangle $\triangle ABC$ at the perspector point X_{10489}.*

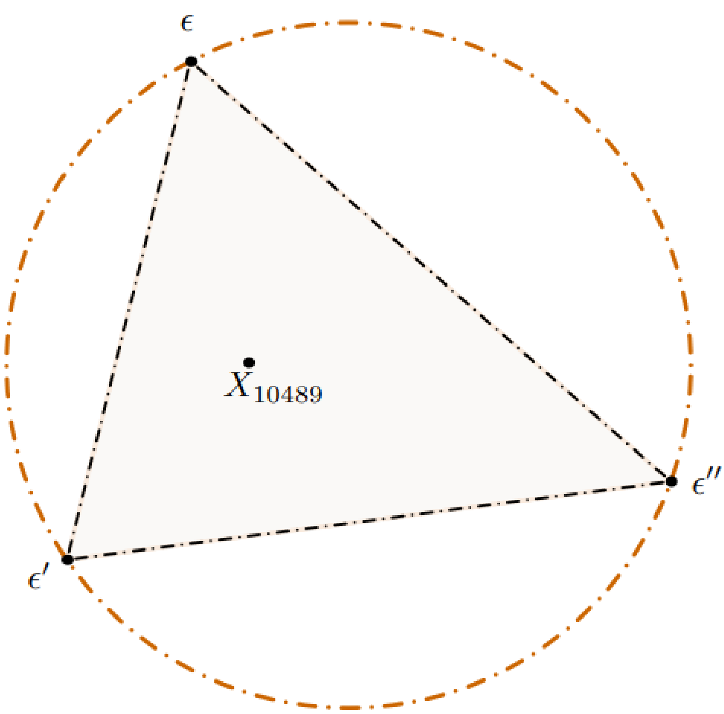

Figure 2. Illustration of triangle $\varepsilon\varepsilon'\varepsilon''$ on the Miyamoto-Moses circle

Theorem 1.7 *Let $\gamma = \gamma_a \cap \omega$, $\gamma' = \gamma_b \cap \omega$, and $\gamma'' = \gamma_c \cap \omega$ be points contained* *within the Miyamoto-Moses circle. Then,* $\triangle\gamma\gamma'\gamma''$ *is in perspective with the intouch triangle or contact triangle (see [7]) at the perspector point* X_{10489}.

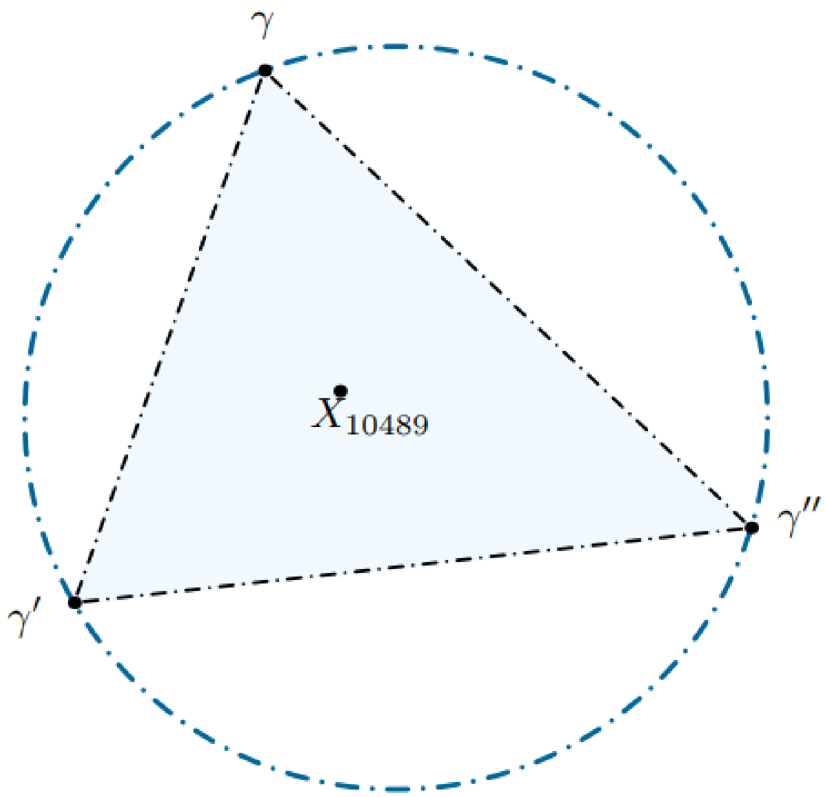

Figure 3. The triangle $\gamma\gamma'\gamma''$ *on the Miyamoto-Moses circle is illustrated.*

Remark 1.8 The point X_{10489} is defined in [6] as the perspector between the mid-arc triangle (the triangle whose vertices are the intersections of the internal angle bisectors with the incircle) and the reference triangle ABC.

II. INSIGHTS LINKED TO THE MIYAMOTO-MOSES-APOLLONIUS CIRCLE

After a meticulous examination of the Miyamoto-Moses circle discovery, a small instance of this circle, as found by Keita Miyamoto, comes to light. This case is exceptionally intriguing, as it gave rise to Apollonian circles, which are described and illustrated further below.

Theorem 2.1 *In a scalene acute triangle ABC, let $M_aM_bM_c$ be its medial triangle. Let Ω_a be the circle centered at M_a and passing through B and C, and define Ω_b and Ω_c cyclically. Inside ABC, let γ_a be the circle externally tangent to lines CA, AB, Ω_a, and define γ_b and γ_c cyclically. Inside ABC, let γ be the circle internally tangent to $\Omega_a, \Omega_b, \Omega_c$. Then there exists a circle Γ that is tangent to the four circles $\gamma, \gamma_a, \gamma_b, \gamma_c$. Here, the circle Γ is named the 1st Miyamoto-Moses-Apollonius circle.*

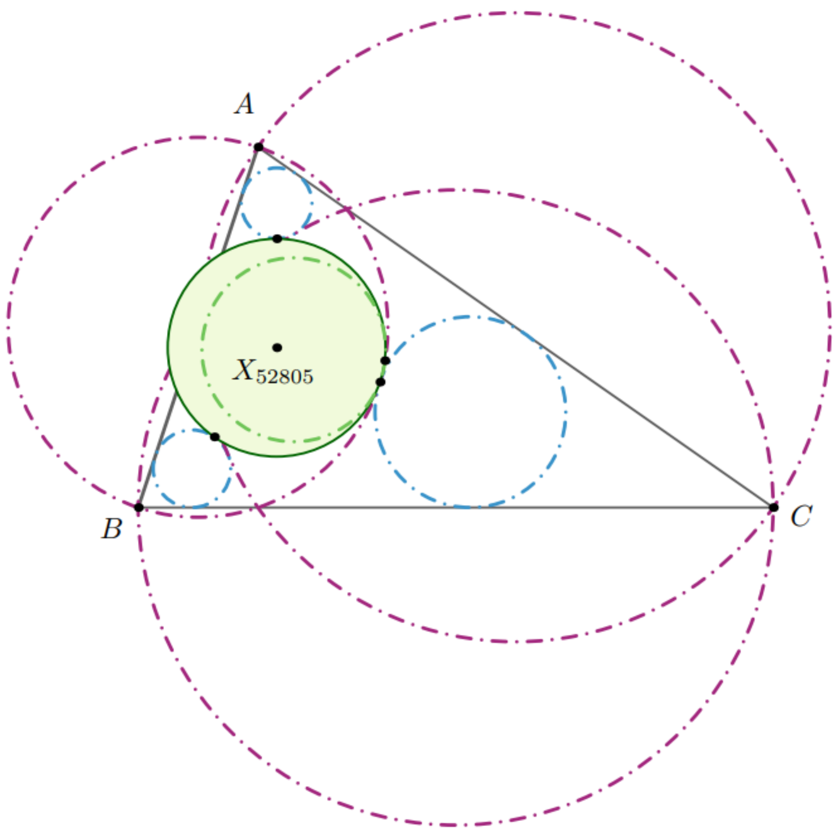

Figure 4. Representation of the 1st Miyamoto-Moses-Apollonius Circle (shaded in green).

In relation to distinctive triangle centers related with this relevant circle, a notable discovery made by Peter Moses in [8] deserves special attention. Further details regarding the definition of this triangle center are presented below.

Definition 2.2 *The point denoted as* X_{52805} *holds the distinction of being the center of the 1st Miyamoto-Moses-Apollonius Circle.*

Furthermore, within the realm of tangent circles, Miyamoto [1] identified a particularly intriguing circle. Its detailed characterization is expounded upon in the subsequent exposition.

Theorem 2.3 *The circle internally tangent to* γ_a, γ_b, *and* γ_c *is named the 2nd Miyamoto-Moses-Apollonius circle.*

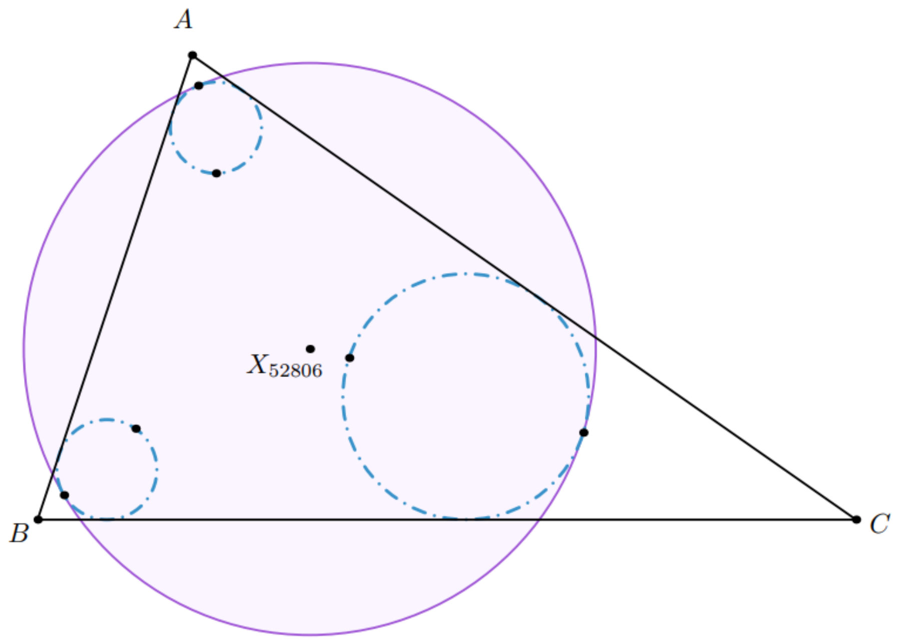

Figure 5. Illustration of the second Miyamoto-Moses-Apollonius Circle (shaded in purple).

From this premise, the formal definition of the point X_{52806} is derived, explicitly delineating that

Definition 2.4 *The point* X_{52806} *is the center of the 2nd Miyamoto-Moses-Apollonius Circle*

As evidenced, the exploration of distinctive triangle centers associated with the pertinent circle has yielded noteworthy outcomes. However, it is worth mentioning that in subsequent works, Moses identifies a new triangle formed by the intersections of the 1st Miyamoto-Moses Apollonius Circle, as presented below.

Theorem 2.5 *The 1st Miyamoto-Moses-Apollonius Triangle is defined as the triangle resulting from the intersections formed by the touchpoints of the 1st Miyamoto-Moses-Apollonius circle* Γ *and the circles externally tangent to the side of the reference triangle, denoted as* γ_a*,* γ_b*, and* γ_c*.*

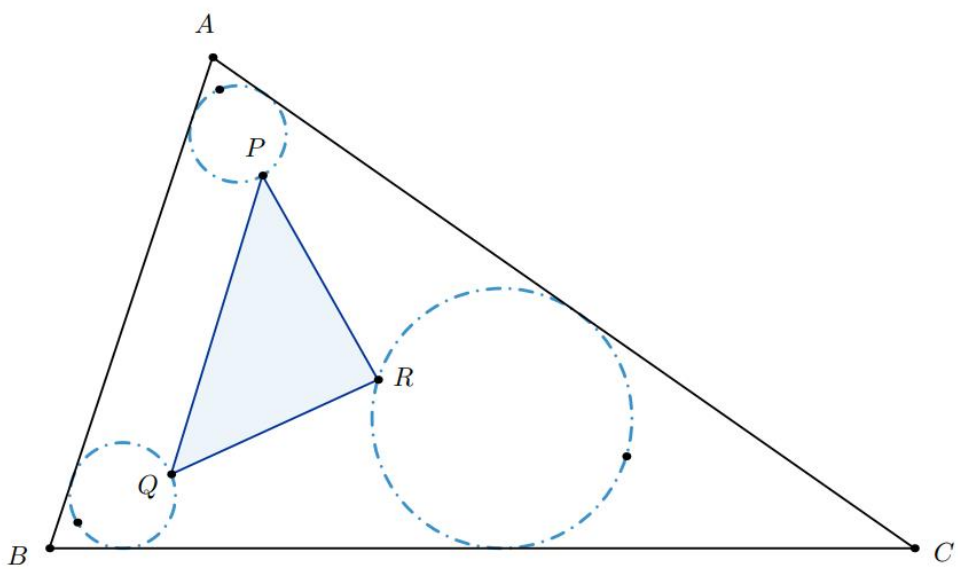

Figure 6. Illustration of the 1st Miyamoto-Moses-Apollonius Triangle (shaded in blue).

The significance of the triangle under consideration arises from the formation of points X_{52811} and X_{52812} as delineated by the following theorems.

Theorem 2.6 *The point X_{52811} is the perspector of the 1st Miyamoto-Moses Apollonius triangle and the Anticomplementary triangle, where the Anticomplementary triangle is defined as the triangle whose medial triangle coincides with the reference triangle.*

Theorem 2.7 *The point X_{52812} serves as the perspector of the 1st Miyamoto-Moses-Apollonius triangle and the Orthic triangle, where the Orthic triangle is characterized as the Cevian Triangle of the Orthocenter.*

These geometric relationships contribute to the mathematical significance of the triangle, providing connections with other notable triangles. In the next sections, attention shall be directed toward the examination of novel findings related to a distinct circle category.

HISTORY OF THE PSEUDO-INSCRIBED CIRCLES

The pseudo-inscribed circles arise from the 43rd International Mathematical Olympiad (IMO). Specifically, they manifest in the context of the second problem of the competition, which is described as follows.

Problem 2.1 *The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\measuredangle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.*

The conclusion to be proved in this question (a circle that passes through two vertices of a triangle and is tangent to the inscribed circle) is a new geometric configuration discovered recently by H. Zichen & L. Sheng [10].

Hereafter, a presentation shall be provided encompassing both a definition and a property of this novel configuration, with the aim of comprehending the subsequent findings delineated in this scholarly research. referencing the contributions of [9].

Definition 2.8 *A circle that is inscribed in the circumcircle of a triangle and tangent to both sides of the triangle is called a pseudo-inscribed circle.*

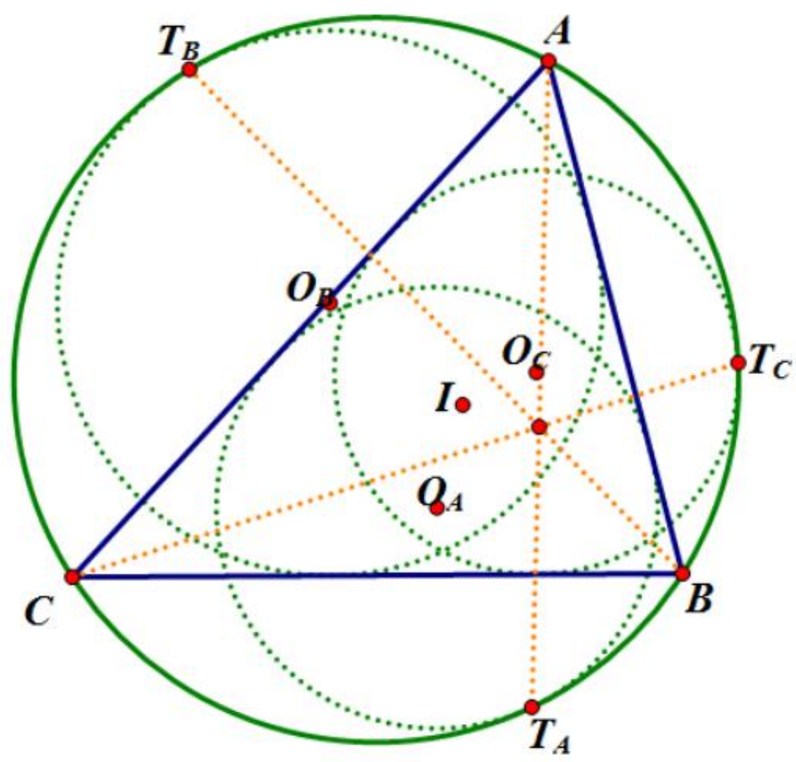

Figure 7. The three circumscribed circles delineated represent pseudo-inscribed circles.

Theorem 2.9 *A triangle has three pseudo-inscribed circles.*

Additionally, pseudo-inscribed circles may experience certain transformations within both sides of the triangles to which they are tangent. To illustrate this, consider the following Lemma originally described in [10].

Lemma 2.10 *Consider a pseudo-inscribed circle at points Z and Y on the respective sides of the triangle in which it is situated. A transformation of this circle is denoted as $Z \rightarrow N$, $Y \rightarrow M$, signifying that the pseudo-inscribed circle now resides at points N and M on these sides.*

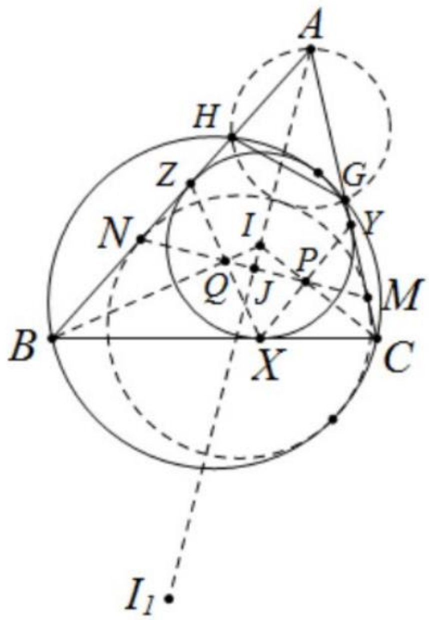

Figure 8. After transformation ($Z \rightarrow N$, $Y \rightarrow M$), it shifts to points N and M.

From the considerations, a discernible property emerges, intrinsic to all previously cited elements, and merits acknowledgment within the context of this study, as elaborated upon below.

Corollary 2.11 *A circle β tangent to the three pseudocircles of a triangle retains its tangency with them as long as the original triangle exists.*

SPECIAL TOPIC STUDY ON PSEUDO-INSCRIBED CIRCLES

After comprehending the preceding information, Miyamoto [1] identified a specific instance in which the circles γ_a, γ_b, and γ_c were to function as pseudo-inscribed circles. Thus, finding a new circle for mathematics in this manner, as illustrated below.

Theorem 2.12 *The 4th Miyamoto-Moses-Apollonius Circle is defined as a circle tangential to the circles γ_a, γ_b, and γ_c.*

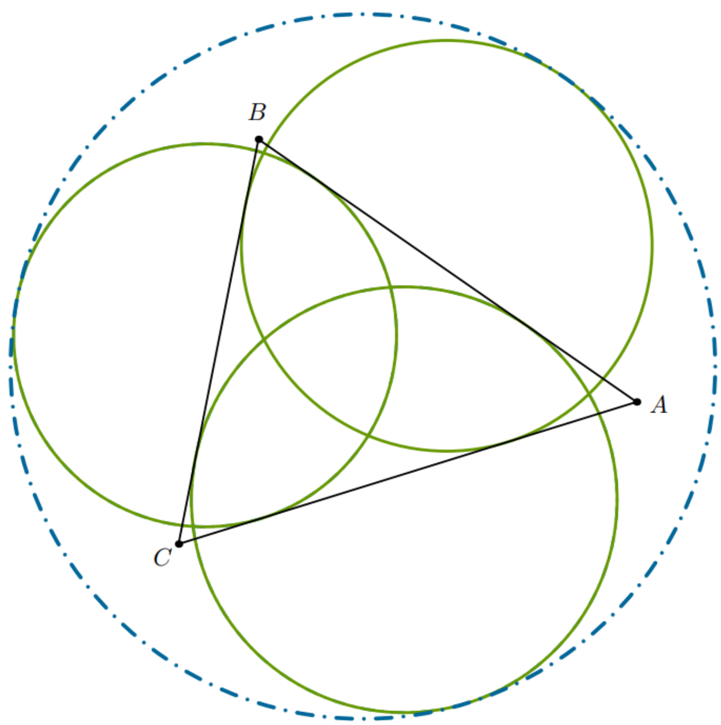

Figure 9. Illustration of the 4th Miyamoto-Moses-Apollonius Circle (dotted in blue).

In this sense, there readily emerges an unspoken theorem regarding pseudo-inscribed circles, to be expounded upon forthwith, as follows:

Theorem 2.13 *A circle in tangential contact with three pseudo-inscribed circles within a triangle gives rise to a pseudo-inscribed triangle with the tangential points.*

Furthermore, in Definition 2.8, a pseudo-inscribed circle ceases to maintain its classification unless it remains tangent to its circumcircle. Nevertheless, this investigation has revealed properties persisting in these circles when are no longer tangent to the circumcircle, prompting the consideration of a novel classification for γ_a, γ_b, and γ_c.

Considering this, a new type of pseudo-inscribed circle is defined as follows:

Definition 2.14 *The Aguilera pseudo-excircle is a type of circle that is tangent to two sides of the given triangle and externally tangent to the 1st Miyamoto-Moses-Apollonius* circle.

Analogously, the following property is defined for Aguilera pseudo-excircles:

Theorem 2.15 *An acute-angled scalene triangle has three Aguilera pseudo-excircles.*

The following diagram is presented to showcase Aguilera pseudo-excircles for explanatory purposes.

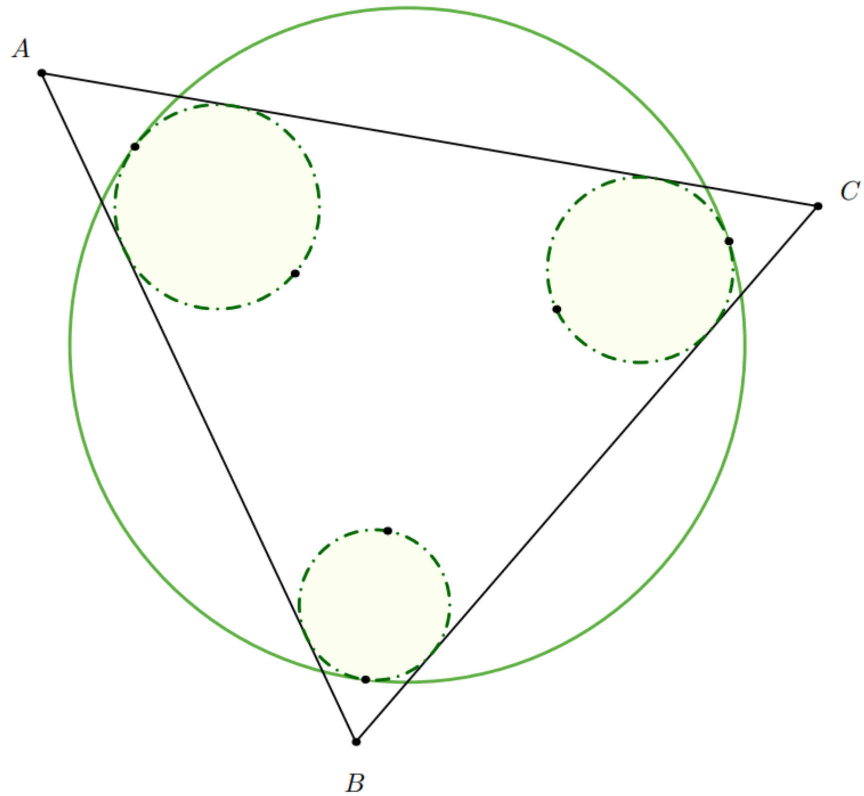

Figure 10. The Aguilera pseudo-excircles of a Triangle (Green Shading)

CHARTING THE RISE OF THE AGUILERA TRIANGLE

Upon careful examination, it is observed that the centers of Aguilera's pseudo-excircles, characterized by the barycentric coordinates $a(-a+b+c)(a+b+c)+2(b+c)S : b(-a+b+c)(a+b+c)-2S : c(-a+b+c)(a+b+c)-2S$.

This intriguing departure from conventional behavior prompts an exploration into the emergence of a novel triangle.

A phenomenon emerges in which a newly defined triangle, related to these barycentric coordinates, exhibits orthological properties in relation to other well-known triangles. This phenomenon is intricately expounded upon by Pavlov in [11]. The triangle in question is referred to as the Aguilera triangle and here, a novel definition of this triangle and some of its properties will be presented within the context of plane geometry.

Definition 2.16 *In an acute-angled scalene triangle ABC, the Aguilera triangle is defined as the triangle formed by the three centers of Aguilera's pseudo-excircles.*

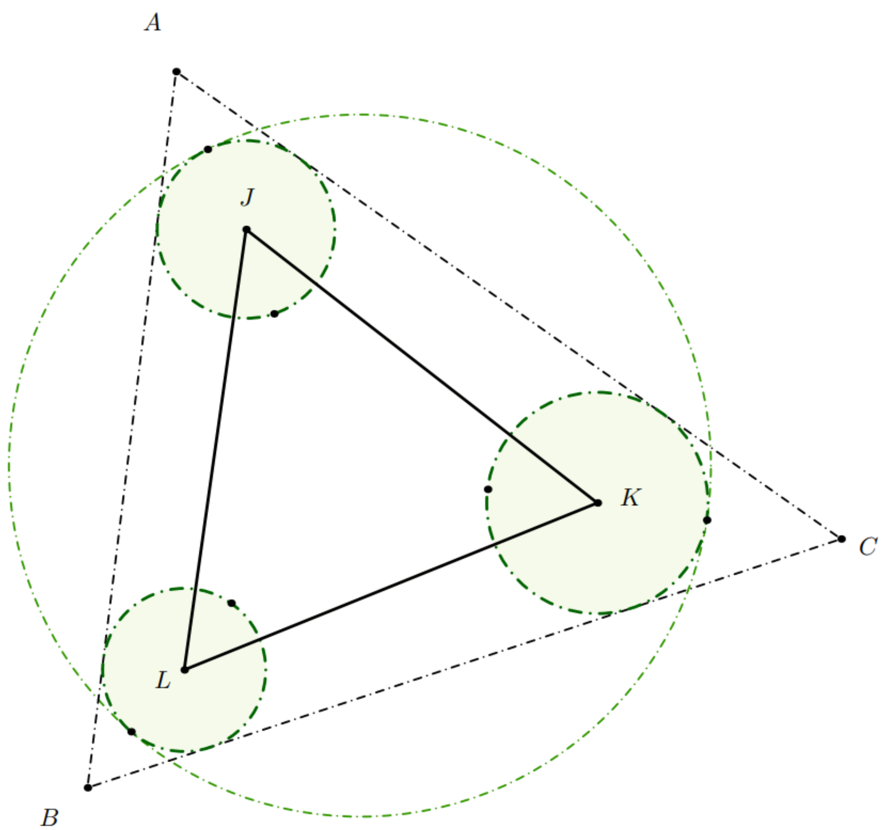

Figure 11. The Aguilera Triangle ($\triangle JKL$)

In Pavlov's research, a comprehensive list of triangles for which the Aguilera triangle is orthologous is provided, as depicted below.

Theorem 2.17 *The Aguilera Triangle is orthological with the following triangles:*

- *Anti-ara*
- *1st Anti-auriga*
- *2nd anti-auriga*
- *5th anti-brocard*
- *2nd anti-circumperp-tangential*
- *2nd anti-Kenmotu-centers*
- *1st anti-Kenmotu-free-vertices*
- *Anti-Lucas(+1) Homothetic*
- *Anti-inner-yff*
- *Ara*
- *2nd Auriga*
- *Anti-ehrmann-mid*
- *Anti-inner-garcia*
- *Anti-outer-grebe*
- *1st anti-kenmotu*
- *2nd anti-kenmotu*
- *1st anti-Kenmotu-centers*
- *2nd anti-Kenmotu-free-vertices*
- *Anti-Lucas(-1) Homothetic*
- *3rd anti-tri-squares-central*
- *Anti-outer-yff*
- *1st auriga*

In Pavlov's research [11], it is suggested that there remain triangles yet to be formally delineated as orthological. Observations and consideration of these triangles would be particularly valuable.

DISCOVERING THE APOLLONIAN CIRCLE PHENOMENON

After comprehending the preceding information, Miyamoto [8] identified a specific instance in which the circles Ω_a, Ω_b, and Ω_c were to function as Apollonian circles. Thus, finding a new circle for mathematics in this manner, as illustrated below.

Theorem 2.18 *The 3rd Miyamoto-Moses-Apollonius Circle is defined as a circle tangential to the Apollonius circles (Ω_a, Ω_b, and Ω_c).*

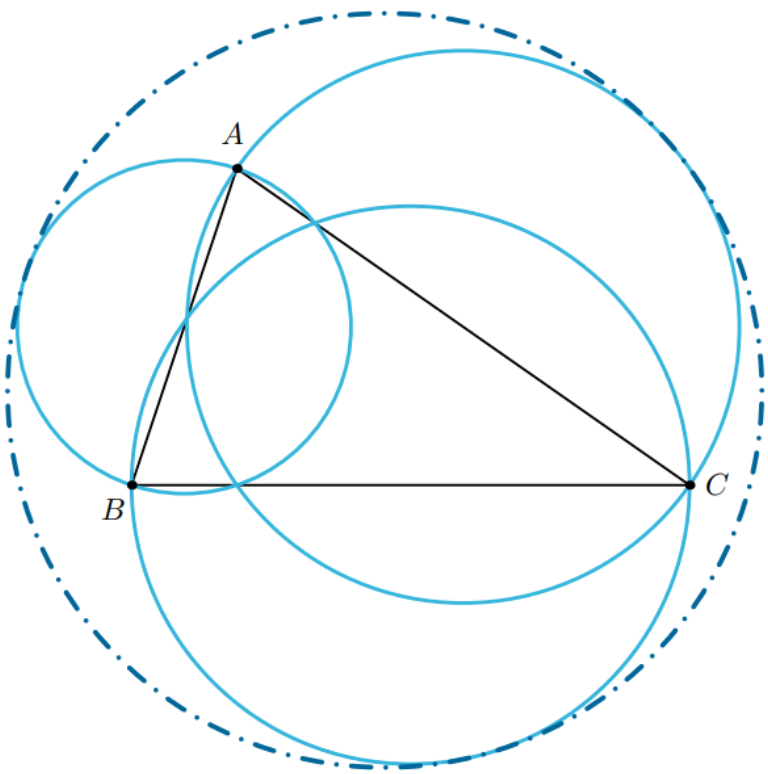

Figure 12.Illustration of the 3rd Miyamoto-Moses-Apollonius Circle (dotted in blue).

Although the Apollonian circles trace their origins to around 200 B.C. (see [12]), the identification of the circle tangential to them had not been documented until Miyamoto's investigative work [8].

For documentary purposes, Miyamoto [8] opted to preserve the triangle formed by the tangency of the 3rd Miyamoto-Moses-Apollonius Circle and the circles of Apollonius due to certain perspective properties outlined in [8]. The definition and illustration of this triangle is presented below.

Definition 2.19 *The 2nd Miyamoto-Moses-Apollonius triangle is the triangle formed by the points of tangency between the 3rd Miyamoto-Moses-Apollonius circle and the Apollonian circles.*

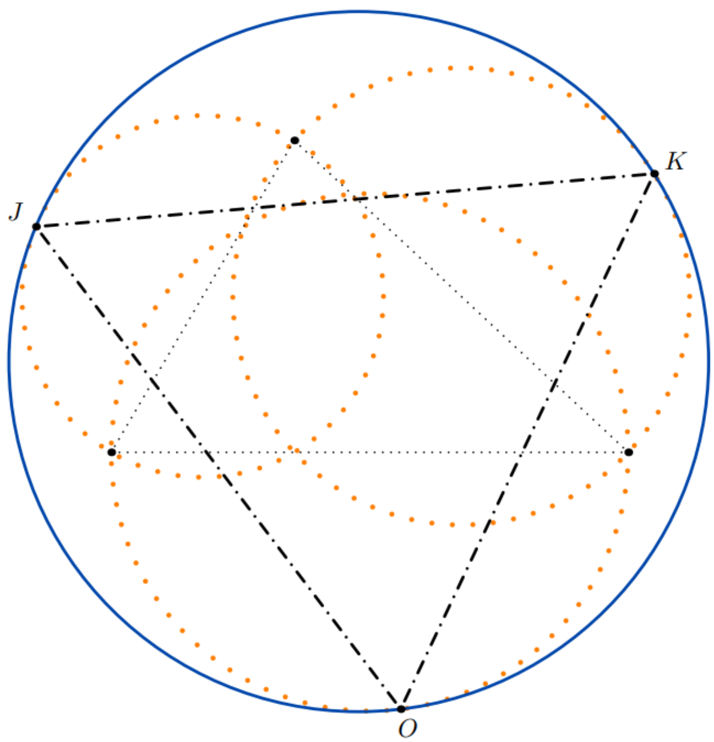

Figure 13. 2nd Miyamoto-Moses-Apollonius Triangle (△JKO).

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